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STABLE STRUCTURES HOMOGENEOUS FOR A FINITE BINARY LANGUAGE†

BY
ALISTAIR H. LACHLAN AND SAHARON SHELAH

Dedicated to the memory of Abraham Robinson on the tenth anniversary of his death

ABSTRACT
Let L be a finite relational language and $\boldsymbol{H}(L)$ denote the class of all countable stable L-structures M for which $\mathrm{Th}(M)$ admits elimination of quantifiers. For $M \in \boldsymbol{H}(L)$ define the rank of M to be the maximum value of $\mathrm{CR}(p,2)$, where p is a complete 1-type over $\varnothing$ and $\mathrm{CR}(p,2)$ is Shelah's complete rank. If L has only unary and binary relation symbols there is a uniform finite bound for the rank of $M \in \boldsymbol{H}(L)$. This theorem confirms part of a conjecture of the first author. Intuitively it says that for each L there is a finite bound on the complexity of the structures in $\boldsymbol{H}(L)$.

1. Introduction

A first-order language will be called *binary* if all its nonlogical symbols are relation symbols which are either unary or binary. All languages considered are relational and finite.

For any finite relational language L let $\boldsymbol{H}(L)$ denote the class of all countable stable L-structures M which are homogeneous for L in the sense of Fraissé, i.e., any isomorphism between finite substructures of M extends to an automorphism of M. In this context to say that M is homogeneous is the same as saying that $\mathrm{Th}(M)$ admits elimination of quantifiers.

There is a notion of rank which is useful in this context. Let M be a structure and $A \subseteq M$ be a nonempty subset of the universe. By $r_M(A)$ we mean the greatest $n < \omega$ if any for which there exist sets $B_\eta \subseteq M$ $(\eta \in {}^{<n}2)$ increasing in η and elements $a_\eta \in A$ $(\eta \in {}^n2)$ such that for distinct $\eta, \zeta \in {}^n2$ and $\sigma \subsetneq \eta$,

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ζ : $\mathrm{tp}(a_\eta \mid B_\sigma) = \mathrm{tp}(a_\zeta \mid B_\sigma)$ iff $\sigma \neq \eta \cap \zeta$. By $\eta \cap \zeta$ we mean the greatest common initial segment of η, ζ. If $n = 0$, then ${}^{<0}2 = \emptyset$ and ${}^{0}2 = \{\lambda\}$ where λ denotes the empty sequence. By convention $r_M(A) = -1$ if $A = \emptyset$. The *rank* $r(M)$ of the structure is defined to be $r_M(M)$. If $M \in \boldsymbol{H}(L)$ for some finite relational language L, then $r(M)$ exists. This notion is related to the complete rank of Shelah [8, p. 55].

The aim of this paper is to prove that, if L is a finite binary language, then there exists $r < \omega$ such that $r(M) < r$ for all $M \in \boldsymbol{H}(L)$. In other words, there is a uniform bound for the rank of a homogeneous L-structure. This result is Theorem 6.1. According to [7] the uniform bound on rank implies that $\boldsymbol{H}(L)$ is the union of a finite number of families such that within each family the isomorphism type of a structure is fixed by its dimensions.

The notion of dimension used is as follows. Let $\boldsymbol{\varphi} = \langle \varphi_0, \varphi_1 \rangle$ be a pair of quantifier-free L-formulas containing at most the variables x_0, x_1, x_2, x_3. Call $\boldsymbol{\varphi}$ a *prenice pair* for $M \in \boldsymbol{H}(L)$ if φ_0, φ_1 define equivalence relations E_0, E_1 on the same subset $D \subseteq M^2$ definable without parameters such that

(i) $E_1 \subsetneq E_0$,

(ii) M^2/E_0 is transitive, i.e. if C_0, C_1 are E_0-classes there exists $\alpha \in \mathrm{aut}(M)$ such that $\alpha(C_0) = C_1$,

(iii) $\{C/E_1 : C \in M^2/E_0\}$ is a family of mutually indiscernible sets, i.e. any $\pi \in \mathrm{perm}(\bigcup\{C/E_1 : C \in M^2/E_0\})$ such that $\pi(C/E_1) = C/E_1$ for each E_0-class C is induced by some $\alpha \in \mathrm{aut}(M)$.

If $\boldsymbol{\varphi}$ is a prenice pair for M, then we say the *$\boldsymbol{\varphi}$-dimension of M* is k where k is the common cardinality of the sets C/E_1 ($C \in M^2/E_0$), and we write $d_M(\boldsymbol{\varphi}) = k$. Otherwise $d_M(\boldsymbol{\varphi})$ is not defined. Observe that once L is fixed there is a finite set Δ of quantifier-free L-formulas containing at most x_0, x_1, x_2, x_3 such that every quantifier-free L-formula containing at most x_0, x_1, x_2, x_3 is logically equivalent to a member of Δ. Thus given the finite relational language L the number of possible pairs $\boldsymbol{\varphi}$ is essentially finite.

One of the corollaries of our main result is that, if L is a finite binary language, there exists $m < \omega$ such that, if $M_i \in \boldsymbol{H}(L)$ $(i < m)$, then there exist distinct $i, j < m$ such that either $M_i \simeq M_j$ or $d_{M_i} \neq d_{M_j}$. For further information we refer the reader to [7].

In [7] it was conjectured that for any finite relational language L there is a uniform bound on $r(M)$ for $M \in \boldsymbol{H}(L)$. This paper settles the conjecture positively for the case of binary languages. More recently Cherlin and the first author have proved the conjecture in general, see [1]. That argument relies on the classification of finite simple groups.

The reason why binary languages are comparatively easy to handle is that, if L is binary, $M \in \boldsymbol{H}(L)$, $A \subseteq M$, and $\bar{b}_0, \ldots, \bar{b}_{n-1} \in M$, then $\mathrm{tp}(\bar{b}_0, \ldots, \bar{b}_{n-1} \mid A)$ is determined by $\mathrm{tp}(\bar{b}_0 \mid A), \ldots, \mathrm{tp}(\bar{b}_{n-1} \mid A)$, and $\mathrm{tp}(\bar{b}_0, \ldots, \bar{b}_{n-1})$. In particular, if A is finite and B is the solution set in M of a complete 1-type over A, then $B \in \boldsymbol{H}(L)$ when B is regarded as an L-structure. Such easy consequences of binariness will be used repeatedly in §4 and §5 without special mention.

The rest of the paper is organized as follows. In §2 are explained most of the notation and terminology used in later sections. In §3 are stated some results about $\boldsymbol{H}(L)$ for L any finite relational language and a "normalization" lemma (3.9) which is Lemma 3 of [5] in a new guise. In §4 we start gathering some information about the behaviour of the rank function $r_M(A)$ (defined in the Introduction) for the case in which $M \in \boldsymbol{H}(L)$ and L is binary. One of the important points here (4.5) is that for fixed L, if $M \in \boldsymbol{H}(L)$, $A \subseteq M$ is finite, and N is the solution set in M of a complete 1-type over A, then $r_M(N)$, the rank of N as a subset of M, can be bounded in terms of $r(N)$, the rank of N as a structure in its own right. In §5 we first prove Lemma 5.1 which is the crucial link in the chain of reasoning which yields the main theorem. This result says that, if L is binary and $M \in \boldsymbol{H}(L)$ is sufficiently large and transitive (i.e. there is only one 1-type over $\varnothing$), then either M is trivial (meaning that M itself is an indiscernible set), or M has a nontrivial 0-definable equivalence relation, or there exist B, $N \subseteq M$ such that $|B| = 2$, N is the solution of a complete 1-type over B, N is large, and $\mathrm{Th}(N)$ has fewer 2-types than $\mathrm{Th}(M)$. This opens the way for us to prove the main theorem for transitive structures by induction on the number of 2-types. The other lemma in §5 is rather technical and its net effect, in the context of 5.1, is to leave us in the same position as if we had proved 5.1 with $|B| = 1$ rather than $|B| = 2$. In §6 various lemmas from §4 and §5 are used to deduce the main theorem.

The key step (5.1) in the proof of the main theorem is due to the second author. The other ideas in the paper come from the first author many of them stemming naturally from [7].

2. Notation and terminology

Relational structures will usually be denoted by M, N possibly with subscripts and superscripts. We shall not distinguish notationally between a structure and its universe. We use $|M|$ to denote the cardinality of M. Subsets of the universe of a structure M will normally be denoted by $A, B, C, \ldots$ possibly with subscripts and superscripts. However, sometimes $A, B, C, \ldots$ will denote not only the subsets of M but also the corresponding substructures. A substructure $M \subseteq N$ is called *full* if, for every 0-definable relation R on N, the restriction of R to M is 0-definable on M. For a binary relation R, $\mathrm{fld}(R)$ denotes $\{a : \exists x(\langle a, x\rangle \in R$ or $\langle x, a\rangle \in R)\}$.

An *extension by definitions* of M is a structure obtained as follows. Let $k < \omega$ and E be a 0-definable equivalence relation with $\mathrm{fld}(E) \subseteq M^k$. Let M^* be the structure with universe $M \dot{\cup} (M^k/E)$, let the relation symbols of M have the same interpretation in M^* as in M, and let M^* have a new $(k+1)$-ary relation symbol R whose interpretation is

$$\{\langle a_0, \ldots, a_{k-1}, b\rangle : a_0, \ldots, a_{k-1} \in M,\ b = \langle a_0, \ldots, a_{k-1}\rangle/E\}.$$

This process of adjoining new elements can be repeated a finite number of times. If $|M| > 1$, any structure obtained in a finite number of steps will be a full 0-definable substructure of one obtained in one step and M may be held fixed. Thus where it is convenient we can always suppose that an extension by definitions is a one-step one.

Throughout the paper M^* denotes an extension by definitions of M which is sufficiently comprehensive to contain all the imaginary elements of M (see [8, III, §6]) that we need in the particular context. If $a \in M^*$, let $[a]$ denote the least $k < \omega$ such that there exists a formula $\varphi(x, \bar{y})$ in the language of M^* and a k-tuple $\bar{b} \in M$ such that $\varphi(x, \bar{b})$ has a as a unique solution.

By M^k we denote the structure with universe M^k whose given relations are all those relations on M^k which are 0-definable in M. Note that M can be identified with the diagonal of M^k if we wish. If E is a 0-definable equivalence relation with $\mathrm{fld}(E) \subseteq M^k$, by M^k/E we mean a structure whose universe is the set of E-classes and whose 0-definable relations are all those induced by relations on M^k 0-definable in M.

If M is a structure and $A \subseteq M^*$, then by (M, A) we denote the structure obtained from M by adjoining for each $a \in A$, where we suppose $a \in M^k/E$ say, a new k-ary relation symbol whose solution is the E-class a. Note that $\mathrm{aut}(M, A)$, the automorphism group of (M, A), is the pointwise stabilizer of A in $\mathrm{aut}(M)$.

If $M \in \boldsymbol{H}(L)$, by a *basic subset* of M^k we mean one defined by an atomic formula or negated atomic formula of L possibly containing some parameters from M.

All types considered are complete. By $\mathrm{tp}(\bar{b}_0, \ldots, \bar{b}_{n-1} \mid A)$ we mean the complete type over A of the tuple $\bar{b}_0 \cdots \bar{b}_{n-1}$. Sometimes we write $\mathrm{tp}_N(\bar{b}_0, \ldots, \bar{b}_{n-1} \mid A)$ to indicate that we mean the type with respect to the structure N. If $k < \omega$ and $A \subseteq M^*$, let $s_k(M, A)$ denote the set of all complete k-types of M over A and let $S_k(M, A)$ denote the set of solution sets in M^k of types in $s_k(M, A)$. Members of $s_k(M, A)$ will be denoted by p, q possibly with subscripts or superscripts. Members of $S_k(M, A)$ will be denoted by P, Q possibly with subscripts or superscripts.

Let perm(A) denote the set or the group of all permutations of A. If $\alpha \in \text{perm}(A)$, then by fix($\alpha$) we mean $\{a \in A : \alpha(a) = a\}$. By aut($M$) we denote the automorphism group of M. If $\alpha \in \text{aut}(M)$ we regard α as being automatically extended to M^* in the obvious way. The structure M is called *transitive* if for all $a, b \in M$ there exists $\alpha \in \text{aut}(M)$ such that $\alpha(a) = b$. If M is countable and $\aleph_0$-categorical, then M is transitive if and only if $|s_1(M, \emptyset)| = 1$.

By $\mathscr{E}(M)$ we denote the set of all 0-definable equivalence relations E such that $\text{fld}(E) \subseteq M$. Let $\mathscr{C}(M)$ denote the set of all C such that C is an E-class for some E in $\mathscr{E}(M)$. For $C \in \mathscr{C}(M)$ define dp(C, M) to be the greatest $n < \omega$ such that there exists $\langle C_0, \ldots, C_n \rangle$ with $C_0 = M$, $C_n = C$, $C_i \in \mathscr{C}(M)$ $(i \leqq n)$, and $C_{i+1} \subsetneqq C_i$ $(i < n)$.

We will need one of the principal concepts of [7], namely that of a nice family of indiscernible sets. Let $E_0, E_1 \in \mathscr{E}(M^2)$ be such that

(i) $\text{fld}(E_0) = \text{fld}(E_1)$ and $E_1 \subsetneqq E_0$,

(ii) M^2/E_0 is transitive,

(iii) $\mathscr{F} = \{C/E_1 : C \in M^2/E_0\}$ is a family of mutually indiscernible sets, i.e. each $I \in \mathscr{F}$ is indiscernible in M^* over $(\bigcup \mathscr{F}) - I$.

The family $\mathscr{F}$ is called a *prenice family* of indiscernible sets. When $M \in \boldsymbol{H}(L)$ this is exactly the same kind of family which is associated with a prenice pair of formulas by the definition in the Introduction. By the *dimension of* $\mathscr{F}$ denoted $d(\mathscr{F})$ we mean the common cardinality of the members of $\mathscr{F}$. Since the members of $\mathscr{F}$ are subsets of M^* rather than M we say that $\mathscr{F}$ is a prenice family *attached to* M.

Let $I \subseteq M^*$ be an indiscernible set and $A \subseteq M^*$. By $\text{cl}^I(A)$ we mean the least subset $B \subseteq I$ if any such that $2|B| < |I|$ and $I - B$ is indiscernible over $A \cup B$. We call $\text{cl}^I(A)$ the *I-closure of* A.

For a prenice family $\mathscr{F}$, the *$\mathscr{F}$-closure of* A denoted $\text{cl}^{\mathscr{F}}(A)$ is $\bigcup\{\text{cl}^I(A) : I \in \mathscr{F}\}$ provided $\text{cl}^I(A)$ is defined for each $I \in \mathscr{F}$.

Let $\mathscr{F}$ be a prenice family attached to M. We say that $\mathscr{F}$ is a *nice family attached to* M if the following additional conditions are satisfied:

(iv) $\text{cl}^{\mathscr{F}}(\{a\})$ exists for all $a \in M$ and $|\text{cl}^{\mathscr{F}}(\{a\}) \cap I| < |I|/8$ for all $I \in \mathscr{F}$,

(v) if $J \subseteq I \in \mathscr{F}$ and $A = \{a \in M : \text{cl}^{\mathscr{F}}(\{a\}) \cap J = \emptyset\}$, then J is strongly indiscernible over $A \cup ((\bigcup \mathscr{F}) - J)$.

For future reference we mention two results about nice families. The first is 11.1 of [7].

LEMMA 2.1. *If $\mathcal{F}$ is a prenice family attached to $M \in \boldsymbol{H}(L)$ and $d(\mathcal{F})$ is large compared with L and $r(M)$, then $\mathcal{F}$ is nice.*

LEMMA 2.2. *Let $M \in \boldsymbol{H}(L)$, $a \in M$, $A \subseteq M$, and $\mathcal{F}$ be a nice family attached to M. Then*

(i) $|\mathrm{cl}^{\mathcal{F}}(\{a\})| < |s_2(M,\emptyset)|$,

(ii) *if* $|A| \leqq d(\mathcal{F})/2|s_2(M,\emptyset)|$, *then* $\mathrm{cl}^{\mathcal{F}}(A)$ *exists,*

(iii) *if* $\mathrm{cl}^{\mathcal{F}}(A)$ *exists, then* $\mathrm{cl}^{\mathcal{F}}(A) = \bigcup\{\mathrm{cl}^{\mathcal{F}}(\{x\}) : x \in A\}$,

(iv) *if* $J \subseteq I \in \mathcal{F}$ *and* $J \cap \mathrm{cl}^{\mathcal{F}}(A) = \emptyset$, *then J is strongly indiscernible over* $A \cup \mathrm{cl}^{\mathcal{F}}(A) \cup ((\bigcup \mathcal{F}) - I)$.

PROOF. (i) As a, b run through M, $|\mathrm{cl}^{\mathcal{F}}(a) \cap \mathrm{cl}^{\mathcal{F}}(b)|$ takes every number $\leqq |\mathrm{cl}^{\mathcal{F}}(a)|$ as a value.

(ii) Suppose $|A| \leqq d(\mathcal{F})/2|s_2(M,\emptyset)|$. Let B denote $\bigcup(\mathrm{cl}^{\mathcal{F}}(\{x\}) : x \in A\}$. For $I \in \mathcal{F}$, $I - B$ is strongly indiscernible over $A \cup B \cup ((\bigcup \mathcal{F}) - I)$ by (v) of the definition of nice family. Therefore $\mathrm{cl}^{\mathcal{F}}(A)$ exists and is $\subseteq B$. Note that $|I \cap B| < d(\mathcal{F})/2$ by (i).

(iii) Let B be defined as above. By the argument for (ii), $\mathrm{cl}^{\mathcal{F}}(A) \subseteq B$. On the other hand, from the definition of $\mathrm{cl}^{I}(A)$, $\mathrm{cl}^{I}(A) \supseteq \mathrm{cl}^{I}(\{x\})$ for each $x \in A$. Hence $\mathrm{cl}^{\mathcal{F}}(A) = B$.

(iv) This follows from the previous part and (v) of the definition of nice family.

If $\mathcal{F}$ is a nice family, by a *complete set of representatives for* $\mathcal{F}$ we mean a set $B \subseteq \bigcup \mathcal{F}$ such that $|B \cap I| = 1$ for each $I \in \mathcal{F}$.

3. Background information

In this section we recall some results from [7] and one from [5] which will be useful in the sequel.

Let M be a structure. By a *quotient structure* of M we mean a structure N such that for some $E_0, E_1 \in \mathscr{E}(M)$

(i) $\mathrm{fld}(E_0) = \mathrm{fld}(E_1)$ and $E_1 \subsetneqq E_0$,

(ii) $N = C/E_1$ for some E_0-class C,

(iii) the 0-definable relations of N are just those induced by the 0-definable relations of M.

We are not specifying any particular language for the quotient structure. Notice that aut(N) is the group of permutations of N induced by aut(M). Obviously M transitive implies that C and N are transitive.

A nice family $\mathscr{F}$ attached to M is called *perfect* if for all $a, b \in M$ and $I, J \in \mathscr{F}$, $|\mathrm{cl}^I(\{a\})| = |\mathrm{cl}^J(\{b\})|$. The set of perfect (nice) families attached to M is denoted $\boldsymbol{F}^p(M)$. If M is stable and $\aleph_0$-categorical, or superstable, then any perfect family attached to M is finite.

The structure M is called *perfect* if there exists a perfect family $\mathscr{F}$ attached to M such that for all $a, b \in M$, $\mathrm{cl}^{\mathscr{F}}(\{a\}) = \mathrm{cl}^{\mathscr{F}}(\{b\})$ iff $a = b$. Notice that a perfect structure is necessarily transitive. For any perfect structure encountered below we will suppose that a particular perfect family $\mathscr{F}$ has been chosen; then for $\mathrm{cl}^{\mathscr{F}}(\{a\})$ we will write $\mathrm{crd}(a)$. We think of "$\mathrm{crd}(a)$" as abbreviating "set of coordinates of a". There are two parameters associated with a perfect structure M: by width(M) we mean $|\mathscr{F}|$, and by index(M) we mean $|I \cap \mathrm{crd}(a)|$ for $I \in \mathscr{F}$ and $a \in M$.

For the rest of this section we fix a finite relational language L. We first recall, without giving proofs, some results from [7]. The first two concern rank and are 4.5 and 5.2 of [7] respectively. The third is 7.3 of [7].

LEMMA 3.1. *If $M \in \boldsymbol{H}(L)$ and $k < \omega$, then $r(M^k)$ can be bounded in terms of k and $r(M)$.*

By saying that "there is a language L' for M such that..." we mean that there is an L'-structure M' with the same universe as M which has the same 0-definable relations such that .... In this situation we may use M to denote M'.

LEMMA 3.2. *Let $M \in \boldsymbol{H}(L)$, $d \in M^*$, and M' denote (M, d).*

(i) *$r(M')$ can be bounded in terms of $[d]$ and $r(M)$.*

(ii) *There is a finite relational language L' for M' such that $M' \in \boldsymbol{H}(L')$ and L' is bounded in terms of $[d]$ and $r(M)$.*

REMARK. Strictly speaking the results of this section should be formulated in terms of a variable language L because we will need to apply 3.2(ii). However, we prefer to suppress the dependence on L of the various bounds whose existence is asserted because this makes the statements of lemmas a little less cumbersome.

LEMMA 3.3. *There is a function $F : \omega^2 \to \omega$ such that, if $M \in \boldsymbol{H}(L)$, $k < \omega$, $E \in \mathscr{E}(M^k)$, $I \subseteq M^k/E$ is 0-definable and $F(r(M), k)$-indiscernible, and $|I| > F(r(M), k)$, then there exists $D \subseteq M$ and $P \in S_k(D)$ such that $|D| \leqq F(r(M), k)$, and the following conditions hold:*

(i) $2|P/E| \geqq |I|$,

(ii) *for each basic $C \subseteq M^k$ either $P \cap a \neq \varnothing$ and $C \cap P \cap a = \varnothing$ for at least half the $a \in I$, or $C \supseteq P \cap a \neq \varnothing$ for at least half the $a \in I$,*

(iii) *if* $n < \omega$ *and* $c_0, \ldots, c_n \in P$ *fall in distinct* E*-classes, then* $\mathrm{tp}(c_0, \ldots, c_n \mid D)$ *depends only on* n.

COROLLARY 3.4. *There exists* $H : \omega^3 \to \omega$ *such that if* $M \in \boldsymbol{H}(L)$, $A \subseteq M$ *is finite,* $I \subseteq M^*$ *is indiscernible and* 0*-definable and* $|I| \geqq H(r(M), [I], |A|)$, *then* $\mathrm{cl}^I(A)$ *exists. In this case* $|\mathrm{cl}^I(A)|$ *can be bounded in terms of* $|A|$.

One of the main results of [7] was 9.4:

TRICHOTOMY THEOREM 3.5. *There exists* $F : \omega \to \omega$ *such that, if* $M \in \boldsymbol{H}(L)$ *and* N *is a transitive quotient structure of* M, *then one of the following three possibilities holds*:

(i) $|N| \leqq F(r(M))$,

(ii) *there is a nontrivial member of* $\mathscr{E}(N)$,

(iii) N *is a perfect structure.*

REMARK 3.6. When N is a perfect structure in the conclusion of 3.5, then

$$\mathrm{width}(N) \cdot \mathrm{index}(N) < |s_2(N, \varnothing)| \leqq |s_2(M, \varnothing)|.$$

The first inequality follows from the fact that $|\mathrm{crd}(a) \cap \mathrm{crd}(b)|$ takes all values $\leqq \mathrm{width}(N) \cdot \mathrm{index}(N)$ as a, b run through N.

The next result did not appear in [7] so we give a proof.

LEMMA 3.7. *There exists* $F : \omega \to \omega$ *such that for every* $G : \omega^2 \to \omega$ *there exists* $H : \omega^2 \to \omega$ *such that, if* $N \in \boldsymbol{H}(L)$, $i < \omega$, *and* $|N| > H(r(N), i)$, *then there exist* $A \subseteq N$, $P \in S_1(N, A)$, *and* $E \in \mathscr{E}(N, A)$ *such that*

(i) $|A| < F(r(N))$,

(ii) $\mathrm{fld}(E) = P$ *and* $|P/E| > G(i, |C|)$ *for each* $C \in P/E$,

(iii) *if* $n < \omega$ *and* $c_0, \ldots, c_n \in P$ *fall in distinct* E*-classes then* $\mathrm{tp}(c_0, \ldots, c_n \mid A)$ *depends only on* n, *and in particular* $|C|$ *is independent of* $C \in P/E$.

PROOF. Suppose that $N \in \boldsymbol{H}(L)$ and that $|N|$ is very large compared with $r(N)$ and i. Without loss we can suppose that N is transitive. Let $\langle C'_j : j < m \rangle$ be a maximal sequence such that for each $j < m$, C'_j is an E'_j-class for some $E'_j \in \mathscr{E}(N)$, and $C'_{j+1} \subsetneqq C'_j$ for each $j < m - 1$. Clearly m is bounded in terms of L, so we can choose $j < m - 1$ such that $|C'_j|$ is very large compared with $|C'_{j+1}|$, $r(N)$, and i. By 3.5, C'_j/E'_{j+1} is a perfect structure which we denote by D. Let E_0, E_1 be as in the definition of perfect structure, C_0 be an E_0-class, and I denote C_0/E_1. Let $a_0, a_1 \in C'_j$ be chosen such that, if $b_0 = a_0/E'_{j+1}$ and $b_1 = a_1/E'_{j+1}$, then

$$|\mathrm{crd}(b_0) - \mathrm{crd}(b_1)| = 1 \quad \& \quad \mathrm{crd}(b_0) - \mathrm{crd}(b_1) \subseteq I.$$

Now I is an imaginary element of N and is an indiscernible set. Since $|C'_j|$ is large compared with $|C'_{j+1}|$, $r(N)$, and i, so is $|I|$, because the width and index of D are bounded by 3.6. By 3.4, $\mathrm{cl}^I(\{a_0, a_1\})$ exists and is bounded in terms of L. There exists $P_0 \in S_1(N, \{a_0, a_1\})$ such that $P_0 \subseteq C'_j$ and for all $b \in P_0/E'_{j+1}$

$$\mathrm{crd}(b_0) \cap \mathrm{crd}(b_1) \subseteq \mathrm{crd}(b) \quad \& \quad \mathrm{crd}(b) \cap I \not\subseteq \mathrm{cl}^I(\{a_0, a_1\}).$$

If $a \in P_0$, then a/E'_{j+1} has a unique coordinate in $I - \mathrm{cl}^I(\{a_0, a_1\})$ and for every $c \in I - \mathrm{cl}^I(\{a_0, a_1\})$ there exists $a \in P_0$ such that $c \in \mathrm{crd}(a/E'_{j+1})$. Let $E_2 = E'_{j+1} \cap (P_0)^2$. Then for all $a, a' \in P_0$

$$aE_2a' \equiv \mathrm{crd}(a/E'_{j+1}) = \mathrm{crd}(a'/E'_{j+1}).$$

Both P_0 and E_2 are $\{a_0, a_1\}$-definable. There is an $\{a_0, a_1\}$-definable bijection between P_0/E_2 and $I - \mathrm{cl}^I(\{a_0, a_1\})$.

Applying 3.3 to the structure $(N, \{a_0, a_1\})$, the set P_0, the equivalence relation E_2, and $I = P_0/E_2$, we obtain $A \subseteq N$, $P \in S_1(N, A)$, and $E = E_2 \cap P^2$ such that $\{a_0, a_1\} \subseteq A$, $|A|$ is bounded in terms of $r(N)$, $P \subseteq P_0$, $2|P/E| \geqq |P_0/E_2|$, and (iii) holds.

Since L is fixed, $|D|$ can be bounded in terms of $|I|$, hence in terms of $|P_0/E_2|$, and hence in terms of $|P/E|$. If C is an E-class, then $|C| \leqq |C'_{j+1}|$. Thus by choosing j such that $|C'_j|$ is sufficiently large compared with $|C'_{j+1}|$ and i we shall ensure that $|P/E| > G(i, |C|)$. (As noted above we also need $|C'_j|/|C'_{j+1}|$ large compared with $r(N)$ in order to apply 3.3.) This is sufficient to prove the lemma.

The next lemma follows from 5.2 and 11.1 of [7].

LEMMA 3.8. *There exists $F : \omega \to \omega$ such that, if $N \in \boldsymbol{H}(L)$, $E \in \mathscr{E}(N)$, C is an E-class, $\mathscr{F} \in \boldsymbol{F}^p(C)$, and $d(\mathscr{F}) \geqq F(r(N))$, then $\mathscr{F}$ is a nice family attached to $(N, \{C\})$.*

The final result of this section is a variant of Lemma 3 of [5]. Since the idea of the previous proof is adequate we give no proof here.

LEMMA 3.9. *Let $R(x, y)$ be a binary relation symbol and $n < \omega$. There is a first-order formula $\varphi_n(x)$ whose only nonlogical symbols are R and $=$ such that, if $m = n^{2^{n-1}}$, then*

$$\forall y_0 \forall y_1 (\exists x_0 R(x_0, y_0) \wedge \exists x_1 R(x_1, y_1) \to \exists^{<n} x (R(x, y_0) \wedge \neg R(x, y_1)))$$

implies

$$\forall y (\exists x R(x, y) \to \exists^{<m} x (R(x, y) \wedge \neg \varphi_n(x)) \wedge \exists^{<m} x (\varphi_n(x) \wedge \neg R(x, y))).$$

REMARK 1. We think of R as defining the family

$$\mathcal{R} = \{\{a \in M : M \models R(a, b)\} : b \in M \models \exists x R(x, b)\}$$

of subsets of whatever structure M we are concerned with. If the difference of any two members of $\mathcal{R}$ has cardinality $< n$, then $\varphi_n(M)$ satisfies $|X - \varphi_n(M)| < m$ and $|\varphi_n(M) - X| < m$ for all $X \in \mathcal{R}$. The significance of the lemma lies in the absence of parameters from $\varphi_n(x)$.

REMARK 2. Clearly the lemma is equally valid when we replace y by $\bar{y}$ and $R(x, y)$ by any formula $\theta(x, \bar{y})$.

4. Rank in binary structures

Here we establish some properties of rank in binary structures. Except for 4.1 we do not know whether the results of this section hold when the language L has arity > 2.

We fix a finite binary language L for the whole section. Where number-theoretic functions F, G are mentioned below, they have L as a hidden parameter.

The upshot of the section (4.8) is that there exists $F : \omega^3 \to \omega$ with the property: if r_0, $m < \omega$ and $r(M) < r_0$ for all transitive $M \in \boldsymbol{H}(L)$ such that $|s_2(M, \emptyset)| < m$, then for any transitive $M \in \boldsymbol{H}(L)$ with $|s_2(M, \emptyset)| = m$ and any $a \in M$ we have

$$r(M) < F(r_0, m, \min\{r(P) : P \in S_1(M, \{a\})\}).$$

From this and the two lemmas of §5 the main theorem (6.1) for transitive structures follows easily. The main theorem is reduced to the case of transitive structures by 4.1 and 4.5.

We first observe that the definition of rank can be given a slightly simpler form since we are assuming the language is binary. If $M \in \boldsymbol{H}(L)$ and $A \subseteq M$, then $r_M(A) \geqq n + 1$ if and only if there exist $a \in M$ and distinct $B_0, B_1 \in S_1(M, \{a\})$ such that $r_M(A \cap B_i) \geqq n$ for $i < 2$. Together with $r_M(\emptyset) = -1$ and $r_M(A) \geqq 0$ for all $A \neq \emptyset$, this is enough to fix $r_M(A)$.

LEMMA 4.1. *If $M \in \boldsymbol{H}(L)$ and $A_0, A_1 \subseteq M$, then*

$$r_M(A_0 \cup A_1) \leqq r_M(A_0) + r_M(A_1) + 1.$$

PROOF. We proceed by induction on $r_M(A_0)$. If $A_0 = \emptyset$ or $A_1 = \emptyset$ the result is true by inspection. Let $a \in M$ and B_0, B_1 be distinct members of $S_1(M, \{a\})$.

There exists $j < 2$ such that $r_M(A_0 \cap B_j) < r_M(A_0)$ and so by the induction hypothesis we have

$$r_M((A_0 \cup A_1) \cap B_j) \leqq r_M(A_0 \cap B_j) + r_M(A_1 \cap B_j) + 1 \leqq r_M(A_0) + r_M(A_1).$$

Since $a \in M$ is arbitrary, we have the desired conclusion.

If $M_0, M_1 \in \boldsymbol{H}(L)$ and $k < \omega$, we say that $s_k(M_0, \emptyset)$ and $s_k(M_1, \emptyset)$ *have a type in common* if there exist $p_0 \in s_k(M_0, \emptyset)$ and $p_1 \in s_k(M_1, \emptyset)$ such that p_0 and p_1 contain the same quantifier-free formulas.

Let $M_0, M_1 \in \boldsymbol{H}(L)$, M_0 be transitive, and $s_2(M_0, \emptyset)$ and $s_2(M_1, \emptyset)$ have at most the trivial 2-type in common. There exist $M \in \boldsymbol{H}(L)$, unique up to isomorphism, $E \in \mathscr{E}(M)$, and a bijection $f: M/E \to M_0$ such that for any E-inequivalent a, b, $\mathrm{tp}_M(a, b)$ and $\mathrm{tp}_{M_0}(f(a/E), f(b/E))$ contain the same quantifier-free formulas and such that each E-class is isomorphic to M_1. By $M_0[M_1]$ we denote any L-structure isomorphic to M. If $s_2(M_0, \emptyset)$ and $s_2(M_1, \emptyset)$ have a non-trivial type in common, we can still form a wreath product $M_0[M_1]$. However, if we define it as an L-structure, then the copies of M_1 will not necessarily be the classes of a 0-definable equivalence relation, nor is there any reason to suppose that $M_0[M_1] \in \boldsymbol{H}(L)$ in this case.

LEMMA 4.2. *If $M_0, M_1 \in \boldsymbol{H}(L)$ have no nontrivial 2-type in common, then $r(M_0[M_1]) \leqq r(M_0) + r(M_1)$.*

PROOF. Let M be the particular copy of $M_0[M_1]$ referred to above. Let C be an E-class. For any $X \subseteq C$, $r_M(X) = r_C(X)$, because for any $a, b \in C$ and $B \subseteq M$, $\mathrm{tp}(a \mid B) = \mathrm{tp}(b \mid B)$ iff $\mathrm{tp}(a \mid B \cap C) = \mathrm{tp}(b \mid B \cap C)$. For $A \subseteq M_0$ let $A[M_1]$ denote $\bigcup f^{-1}(A)$. If $B \subseteq M$ is finite and $P \in S_1(M, B)$, then either $P = A[M_1]$ for some $A \in S_1(M_0, f(B/E))$ or $P \in S_1(C, B \cap C)$ for some $C \in B/E$. By induction on $r_{M_0}(A)$ we can see that, if $A \subseteq M_0$ and $A[M_1] \in S_1(M, B)$ for some finite $B \subseteq M$, then $r_M(A[M_1]) \leqq r_{M_0}(A) + r(M_1)$. Taking $A = M_0$ we are done.

LEMMA 4.3. *Let $M \in \boldsymbol{H}(L)$, $A \subseteq M$ be finite, and $M' = (M, A)$.*

(i) *If $B \subseteq M$, then $r_{M'}(B) \leqq r_M(B)$.*

(ii) *If $B \subseteq P \in S_1(M, A)$, then $r_{M'}(B) = r_M(B)$.*

PROOF. (i) If $P \in S_1(M', D)$ for some finite $D \subseteq M$, then $P \in S_1(M, A \cup D)$. This is enough.

(ii) Suppose $B \subseteq P \in S_1(M, A)$. If $Q \in S_1(M, D)$ for some finite $D \subseteq M$, then $Q \cap P \in S_1(M', D)$. This is enough.

In the next lemma we shall use the notation $\mathrm{cl}^{\mathscr{F}}_M(B)$ with a special meaning which we now explain. A structure $M \in \boldsymbol{H}(L)$ is given as are a finite set $A \subseteq M$, $N \in S_1(A)$, $E \in \mathscr{E}(N)$, and E-class C, $\mathscr{F} \in \boldsymbol{F}^p(C)$, and a finite set $B \subseteq M$. By $\mathrm{cl}^{\mathscr{F}}_M(B)$ we mean the least $D \subseteq \bigcup \mathscr{F}$ if any with the following properties:

(i) for each $I \in \mathscr{F}$, $2|D \cap I| < |I|$,

(ii) if $\pi \in \mathrm{perm}(\bigcup \mathscr{F})$, $\pi(I) = I$ for each $I \in \mathscr{F}$, and $D \subseteq \mathrm{fix}(\pi)$, then there exists $\alpha \in \mathrm{aut}(M)$ such that α induces π and $A \cup B \cup \{C\} \cup D \subseteq \mathrm{fix}(\alpha)$.

Lemma 4.4. *There exist $F : \omega^2 \to \omega$ and $G : \omega \to \omega$ such that, if $M \in \boldsymbol{H}(L)$, $N \in S_1(A)$ for some finite $A \subseteq M$, $C \in \mathscr{E}(N)$, $B \subseteq M$ is finite, $\mathscr{F} \in \boldsymbol{F}^p(C)$, and $d(\mathscr{F}) \geqq F(|B|, r(N))$, then $\mathrm{cl}^{\mathscr{F}}_M(B)$ exists and $|\mathrm{cl}^{\mathscr{F}}_M(B)| \leqq G(|B|)$.*

Proof. Since $\mathscr{F} \in \boldsymbol{F}^p(C)$, $k = |\mathscr{F}|$ is bounded by $|s_2(C, \varnothing)|$ by 3.6 and hence is bounded in terms of L. Supposing that $d(\mathscr{F})$ is very large compared with $|B|$ and $r(N)$ we shall prove that $\mathrm{cl}^{\mathscr{F}}_M(B)$ exists.

Let $E_0, E_1 \in \mathscr{E}(C^2)$ determine $\mathscr{F}$ and let $D = \mathrm{fld}(E_0) = \mathrm{fld}(E_1)$. Let $\langle D_i : i < k \rangle$ be an enumeration of D/E_0, and I_i denote D_i/E_1. By 3.8, since $d(\mathscr{F})$ is large compared with $r(N)$, $\mathscr{F} = \{I_i : i < k\}$ is a nice family attached to $(N, \{C\})$. Below $\mathrm{cl}^{\mathscr{F}}_N(\ )$ is to be understood as $\mathrm{cl}^{\mathscr{F}}(\ )$ relative to the structure $(N, \{C\})$.

From 3.1, $r(N^2)$ is small compared with $d(\mathscr{F})$. From 3.2, the rank and language of $(N, \{C\})$ as a homogeneous structure can be bounded in terms of L and $r(N)$. Therefore, by 2.2, $\mathrm{cl}^{\mathscr{F}}_N(X)$ exists for any $X \subseteq N$ with $|X|$ small compared with $d(\mathscr{F})$; moreover, when $|X|$ is small, so is $|\mathrm{cl}^{\mathscr{F}}_N(X)|$.

Call $\langle X, \boldsymbol{P} \rangle$ a *good pair* if the following conditions are satisfied:

(i) $X \subseteq N$, $|X| \leqq k \cdot r(N^2)$,

(ii) $\boldsymbol{P} = \langle P_i : i < k \rangle$, $P_i \in S_2(N, X)$ $(i < k)$,

(iii) for $a \in I_i$, $P_i \cap a \neq \varnothing$ iff $a \notin \mathrm{cl}^{\mathscr{F}}_N(X)$.

Define $m(X, \boldsymbol{P}) = \Sigma\{r_{N^2}(P_i \cap a_i) : i < k\}$, where $a_i \in I_i - \mathrm{cl}^{\mathscr{F}}_N(X)$ $(i < k)$. (Since $P_i \in S_2(N, X)$ and $a_i \notin \mathrm{cl}^{\mathscr{F}}_N(X)$, $r_{N^2}(P_i \cap a_i)$ does not depend on the choice of a_i.)

Fix a good pair $\langle X, \boldsymbol{P} \rangle$ with

$$(\#) \qquad m(X, \boldsymbol{P}) < k \cdot r(N^2) - |X|$$

so as to minimize $m(X, \boldsymbol{P})$. We claim:

(iv) if $d \in N$, $Q \subseteq N^2$ is a $\{d\}$-definable basic subset in the sense of N, $i < k$, and $a \in I_i - \mathrm{cl}^{\mathscr{F}}_N(X \cup \{d\})$, then either $Q \supseteq P \cap a$ or $Q \cap P \cap a = \varnothing$.

By inspection $\boldsymbol{P}$ can be chosen so that $\langle \varnothing, \boldsymbol{P} \rangle$ is a good pair satisfying $(\#)$. Thus $\langle X, \boldsymbol{P} \rangle$ can be chosen as stipulated. Towards a contradiction suppose (iv) fails through $d \in N$, $Q \subseteq N^2$, and $i < k$.

From the definition of $\mathrm{cl}^{\mathscr{F}}_N$, for $j < k$, every $a \in I_j - \mathrm{cl}^{\mathscr{F}}_N(X \cup \{d\})$ realizes the same type over $X\cup\{d,C\}$ and hence over $\{X,d\}$. Also, $\mathrm{cl}^{\mathscr{F}}_N(X\cup\{d\})$ is $X\cup\{d\}$-definable in $(N,\{C\})$. Since $\{C\}$ is $\{x\}$-definable uniformly for $x\in\bigcup\mathscr{F}$, if $P\in S_2(N,X\cup\{d\})$ and $P\cap a\neq\emptyset$ for some $a\in I_i-\mathrm{cl}^{\mathscr{F}}_N(X\cup\{d\})$, then, for all $a\in I_i$, $P\cap a\neq\emptyset$ iff $a\notin\mathrm{cl}^{\mathscr{F}}_N(X\cup\{d\})$. For each $j<k$, $j\neq i$, choose $P'_j\in S_2(X\cup\{d\})$ such that $P'_j\subseteq P_j$ and $P'_j\cap a\neq\emptyset$ for $a\in I_j-\mathrm{cl}^{\mathscr{F}}_N(X\cup\{d\})$. Clearly, $r_{N^2}(P'_j\cap a_j)\leqq r_{N^2}(P_j\cap a_j)$ $(j<k,\ j\neq i)$.

By assumption, for $a\in I_i-\mathrm{cl}^{\mathscr{F}}_N(X\cup\{d\})$, there exist distinct $P',P''\in S_2(N,X\cup\{d\})$ both $\subseteq P$ such that $P'\cap a$ and $P''\cap a$ are both non-empty. One of $r_{N^2}(P'\cap a)$, $r_{N^2}(P''\cap a)$ is less than $r_{N^2}(P_i\cap a)$.

Choosing P'_i appropriately, we obtain a good pair $\langle X\cup\{d\},\boldsymbol{P}'\rangle$ such that $m(X\cup\{d\},\boldsymbol{P}')<m(X,\boldsymbol{P})$. This contradicts the choice of $\langle X,\boldsymbol{P}\rangle$, and so (iv) holds.

From the discussion it is clear that $\mathrm{cl}^{\mathscr{F}}_N(X)=\bigcup\{I_i-(P_i/E_1):i<k\}$. Thus the sets $(D_i\cap P_i)/E_1$ $(i<k)$ are strongly mutually indiscernible over $X\cup\mathrm{cl}^{\mathscr{F}}_N(X)\cup\{C\}$. It is also clear that $|\mathrm{cl}^{\mathscr{F}}_N(X)|$ is small compared with $d(\mathscr{F})$.

Claim 1. *Let $i,j<k$, $a\in D_i\cap P_i$ and $b\in D_j\cap P_j$ such that $a\not\mathrm{E}_1 b$. Then* $\mathrm{tp}(a,b)$ *depends only on* (i,j).

Proof of Claim 1. Towards a contradiction suppose there exists $b'\in D_j\cap P_j$ such that $a\not\mathrm{E}_1 b'$ and $\mathrm{tp}(a,b')\neq\mathrm{tp}(a,b)$. Since $(D_i\cap P_i)/E_1$ and $(D_j\cap P_j)/E_1$ are mutually indiscernible, switching i and j if necessary we choose a, b and b' such that $b\,\mathrm{E}_1\,b'$. Then a splits $P_j\cap(b/E_1)$. If every $a'\in P_i\cap(a/E_1)$ splits $P_j\cap(b/E_1)$, then every $a'\in D_i\cap P_i$ splits $P_j\cap(c/E_1)$ for every $c\in(D_j\cap P_j)-(a'/E_1)$. This contradicts property (iv) of $\langle X,\boldsymbol{P}\rangle$. Thus not every $a'\in P_i\cap(a/E_1)$ splits $P_j\cap(b/E_1)$. Hence for every $c\in(D_i\cap P_i)-(b/E_1)$ some elements of $P_i\cap(c/E_1)$ split $P_j\cap(b/E_1)$ and some do not. Now

$$Q=\{x\in(D_i\cap P_i)-(b/E_1):x\text{ splits }P_j\cap(b/E_1)\}$$

splits $P_i\cap(c/E_1)$ for every $c\in(D_i\cap P_i)-(b/E_1)$. This contradicts property (iv) of $\langle X,\boldsymbol{P}\rangle$ and so completes the proof of Claim 1.

To show that $\mathrm{cl}^{\mathscr{F}}_M(B)$ exists it suffices to show that $\mathrm{cl}^{\mathscr{F}}_M(B')$ exists for some $B'\supseteq B$. Thus there is no loss of generality if we adjoin elements of N to B in order to have $B\cap C\neq\emptyset$ so that C and $\bigcup\mathscr{F}$ become $A\cup B$-definable, and to have $B^2\cap b\neq\emptyset$ for each $b\in\bigcup\{I_i-(P_i/E_1):i<k\}$ so that each singleton $\subseteq I_i-(P_i/E_1)$ is B-definable $(i<k)$. Note that $|B|$ is still small compared with $d(\mathscr{F})$.

Claim 2. *For each $i<k$ there exists $Q_i\in S_2(M,B)$ such that $|(D_i\cap P_i)/E_1-(D_i\cap P_i\cap Q_i)/E_1|$ is bounded in terms of $|B|$.*

Proof of Claim 2. Let $\bar{b}$ enumerate B, $i < k$, and $m_i =_{\text{dfn}} |(D_i \cap P_i)/E_1|$. Let Y be any B-definable subset of M^2. Let $I_i(Y) = (D_i \cap P_i \cap Y)/E_1$ and

$$I'_i(Y) = \{a \in (D_i \cap P_i)/E_1 : a \cap P_i \cap Y = \emptyset\}.$$

Until further notice assume $|I'_i(Y)| \geqq |I_i(Y)|$. If $m_i < \omega$, this is the same as $|I_i(Y)| \leqq m_i/2$. Define $J(\bar{b}) = \bigcup\{I_j(Y) : j < k\}$. Since we arranged for C to be $A \cup B$-definable, $J(\bar{b})$ is $A \cup X \cup B$-definable.

Since $N \cup A$ is A-definable and M is ω-categorical and ω-stable, M is prime over $N \cup A$. Therefore every $\alpha \in \text{aut}(N)$ extends to $\beta \in \text{aut}(M)$ with $A \subseteq \text{fix}(\beta)$. Let $\pi \in \text{perm}((D_i \cap P_i)/E_1)$. There exists $\alpha \in \text{aut}(N)$ inducing π such that

$$X \cup \{C\} \cup ((\bigcup \mathcal{F}) - ((D_i \cap P_i)/E_1)) \subseteq \text{fix}(\alpha).$$

Let $\beta \in \text{aut}(M)$ extend α with $A \subseteq \text{fix}(\alpha)$, and $J(\beta(\bar{b}))$ denote the image of $J(\bar{b})$ under β. Notice that $J(\bar{b}) - J(\beta(\bar{b})) = I_i(Y) - \pi(I_i(Y))$. Since we arranged for C to be $A \cup B$-definable, $|J(\bar{b}) - J(\beta(\bar{b}))|$ is a function of $\text{tp}(\bar{b}^\frown \beta(\bar{b}) \mid A \cup X)$. Since $\text{tp}(\bar{b} \mid A \cup X) = \text{tp}(\beta(\bar{b}) \mid A \cup X)$, $|J(\bar{b}) - J(\beta(\bar{b}))|$ is a function of $\text{tp}(\bar{b}^\frown \beta(\bar{b}))$. (This is one of several places where we use the binariness of the language.) By varying π we see that $|J(\bar{b}) - J(\beta(\bar{b}))|$ takes at least $|I_i(Y)|$ different values. Hence there are $\geqq |I_i(Y)|$ possibilities for $\text{tp}(\bar{b}^\frown \beta(\bar{b}))$, i.e., $|I_i(Y)| \leqq |s_{2|B|}(M, \emptyset)|$.

Similarly, assuming $|I'_i(Y)| \leqq |I_i(Y)|$, we can prove $|I'_i(Y)| \leqq |s_{2|B|}(M, \emptyset)|$. Recall that $|\text{cl}_N^{\mathcal{F}}(X)|$ is small compared with $d(\mathcal{F})$, as is $|B|$. Clearly, $|S_2(M, B)|$ being bounded in terms of $|B|$, we can find $Q \in S_2(M, B)$ such that $|I_i(Q)| \not\leqq |S_{2|B|}(M, \emptyset)|$. The argument above tells us that $|I_i(Q)| \not\leqq |I'_i(Q)|$, i.e., $|I'_i(Q)| \leqq |I_i(Q)|$. This implies that $|I'_i(Q)| \leqq |S_{2|B|}(M, \emptyset)|$. Since $I'_i(Q) = (D_i \cap P_i)/E_1 - (D_i \cap P_i \cap Q)/E_1$, the claim is proved.

For $i < k$, let $Q_i \in S_2(M, B)$ satisfy the conclusion of Claim 2 for D_i and P_i. For each $i < k$, let $f_i : (D_i \cap P_i)/E_1 \to D_i \cap P_i$ be a function such that $f_i(x)/E_1 = x$ $(x \in (D_i \cap P_i)/E_1)$ and $f_i(x) \in Q_i$ $(x \in I_i(Q_i))$. Let $K_i = \text{rng}(f_i \restriction I_i(Q))$ and $R_i = \text{rng}(f_i) - K_i$ $(i < k)$. Let K'_i denote $\bigcup\{\{x_0, x_1\} : \langle x_0, x_1\rangle \in K_i\}$ and R'_i denote $\bigcup\{\{x_0, x_1\} : \langle x_0, x_1\rangle \in R_i\}$. If $a \in (D_i \cap P_i)/E_1$, $b \in (D_j \cap P_j)/E_1$, $a \neq b$, $f(a) = \langle a_0, a_1\rangle$, $f(b) = \langle b_0, b_1\rangle$, and $a_0 = b_0$, then by Claim 1 every pair in $(D_i \cap P_i) \cup (D_j \cap P_j)$ has the same first member. We can make a similar inference if one of $a_0 = b_1$ and $a_1 = b_1$ holds instead of $a_0 = b_0$.

Consider $\pi \in \text{perm}(\bigcup\{K_i : i < k\})$ such that $\pi(K_i) = K_i$ $(i < k)$. From our remarks above there exists $\gamma \in \text{perm}(\bigcup\{K'_i : i < k\})$ induced by π which has no conflict with the identity map on $\bigcup\{R'_i : i < k\}$. Let γ' be the union of γ and the identity map on $A \cup B \cup X \cup \bigcup\{R'_i : i < k\}$. Since L is a binary language, γ' is an elementary map in the context of $M \in \boldsymbol{H}(L)$. Let $\delta \in \text{perm}(A \cup B \cup X \cup \bigcup \mathcal{F})$ be the map induced by γ'. (This is where we use the fact that $\{a\}$ is B-definable for each $a \in (\bigcup \mathcal{F}) - \text{cl}_N^{\mathcal{F}}(X)$, so that δ fixes each such a. Every other $a \in \bigcup \mathcal{F}$ has a representative mapped by γ'.) Since $A \cup B \cup X \cup \bigcup \mathcal{F}$ is $A \cup B \cup X$-definable, M^* is prime over $A \cup B \cup X \cup \bigcup \mathcal{F}$. It follows that δ is induced by some $\alpha \in \text{aut}(M)$. For each $i < k$, α permutes $I_i - ((D_i \cap P_i \cap Q_i)/E_1)$ arbitrarily and fixes the rest of I_i, which is small compared with $d(\mathcal{F})$, pointwise. Therefore $\text{cl}_M^{\mathcal{F}}(B)$ exists.

It is easy to see that $\text{cl}_M^{\mathcal{F}}(\varnothing) = \varnothing$, because M^* is prime over $A \cup \{C\} \cup \bigcup \mathcal{F}$. Now we want to see that $|\text{cl}_M^{\mathcal{F}}(B)|$ can be bounded in terms of $|B|$. The argument is like that for Claim 2. By adjoining one element to B, we ensure that C and hence $\bigcup \mathcal{F}$ are B-definable. For any $j \leqq |\text{cl}_M^{\mathcal{F}}(B)|$, there exists $\alpha \in \text{aut}(N)$ such tht $\alpha(C) = C$ and

$$|\text{cl}_M^{\mathcal{F}}(B) \cap \alpha(\text{cl}_M^{\mathcal{F}}(B))| = j.$$

Let $\beta \in \text{aut}(M)$ extend α such that $A \subseteq \text{fix}(\beta)$. Note that $\text{cl}_M^{\mathcal{F}}(B) \cap \alpha(\text{cl}_M^{\mathcal{F}}(B)) = \text{cl}_M^{\mathcal{F}}(B) \cap \text{cl}_M^{\mathcal{F}}(\beta(B))$ is $A \cup B \cup \beta(B)$-definable. Thus j is determined by $\text{tp}(\bar{b}^\frown \beta(\bar{b}) \mid A)$. But $\text{tp}(\bar{b} \mid A) = \text{tp}(\beta(\bar{b}) \mid A)$ and so, since the language is binary, $\text{tp}(\bar{b}^\frown \beta(\bar{b}) \mid A)$ is determined by $\text{tp}(\bar{b}^\frown \beta(\bar{b}))$. Hence

$$|\text{cl}_M^{\mathcal{F}}(B)| \leqq |S_{2|B|}(M, \varnothing)|$$

which is the required bound.

The previous lemma is only required to obtain:

Lemma 4.5. *There exists $F : \omega \to \omega$ such that, if $M \in \boldsymbol{H}(L)$ and $N \in S_1(A)$ for some finite $A \subseteq M$, then $r_M(N) \leqq F(r(N))$.*

Proof. For a contradiction argument suppose the lemma fails. Let $r_0 < \omega$ be least such that for every $r < \omega$ there exist $M \in \boldsymbol{H}(L)$, finite $A \subseteq M$, and $N \in S_1(A)$ with $r(N) = r_0$ and $r_M(N) > r$. We choose a sequence $\boldsymbol{S} = \langle\langle M_i, A_i, N_i, C_i \rangle : i < \omega\rangle$ such that for all $i < \omega$

(i) $M_i \in \boldsymbol{H}(L)$,
(ii) $A_i \subseteq M_i$ is finite,
(iii) $N_i \in S_1(A_i)$,
(iv) $C_i \in \mathscr{C}(N_i)$,
(v) $r(N_i) = r_0$ and $r_{M_i}(C_i) > i$.

For any $N \in \boldsymbol{H}(L)$ and $C \in \mathscr{E}(N)$ there is an absolute bound on $\text{dp}(C, N)$. Thus we can choose the sequence $\boldsymbol{S}$ such that $d_0 = \text{dp}(C_i, N_i)$ does not depend on

i and is as large as possible. For all sufficiently large i there exists $\mathcal{F}_i \in \boldsymbol{F}^p(C_i)$. Otherwise by 3.5 for infinitely many i there would be $E_i \in \mathscr{E}(C_i)$ with $|C_i/E_i|$ bounded in terms of r_0. By 4.1 this would allow us to increase d_0. Thus we may suppose that for all i there exist $\mathcal{F}_i \in \boldsymbol{F}^p(C_i)$. Let $\boldsymbol{S}'$ denote a sequence

$$\langle\langle M_i, A_i, N_i, C_i, \mathcal{F}_i, B_i\rangle : i < \omega\rangle$$

such that for all $i < \omega$, we have (i)–(v) and also

(vi) $\operatorname{dp}(C_i, N_i) = d_0$,

(vii) $\mathcal{F}_i \in \boldsymbol{F}^p(C_i)$,

(viii) $B_i \subseteq \bigcup \mathcal{F}_i$ is finite,

(ix) $r_M(\{a \in C_i : \operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\}) \supseteq B_i\}) > i$.

Since $C_i \in \boldsymbol{H}(L)$, $|\operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\})|$ is bounded. Hence $|B_i|$ is also bounded. Thus we can suppose that $|B_i| = m$ does not depend on i and that $\boldsymbol{S}'$ is chosen so as to maximize m. If $|\operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\})| = |B_i|$ for $a \in C_i$, then there exists $E_i \in \mathscr{E}(C_i)$ such that

$$\{a \in C_i : \operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\}) \supseteq B_i\}$$

is an E_i-class. Thus if this happened for infinitely many i we could increase d_0, contradiction. So we may assume that for all i

$$|\operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\})| > |B_i| = m \qquad (a \in C_i).$$

Since $\boldsymbol{S}'$ was chosen to maximize m there exists $r_1 < \omega$ such that for all $i < \omega$, if $b \in (\bigcup \mathcal{F}_i) - B_i$, then

$$r_{M_i}(\{a \in C_i : \operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\}) \supseteq B_i \cup \{b\}\}) < r_1.$$

Thus we can choose $r_1 < \omega$ such that for all $i < \omega$

(x) if B'_i is a complete set of representatives for $\mathcal{F}_i$ and $B_i \cap B'_i = \varnothing$, then

$$r_{M_i}(\{a \in C_i : \operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\}) \supseteq B_i \ \&\ \operatorname{cl}^{\mathcal{F}}_{C_i}(\{a\}) \cap B'_i \neq \varnothing)\} < r_1.$$

Because we are dealing with structures homogeneous for a binary language the definition of rank given in the Introduction can be simplified. We can stipulate that B_η in that definition has cardinality $l(\eta)+1$. For any $D \subseteq M$, $r_M(D)$ is the greatest $n < \omega$ if any for which there exist $a(\eta) \in M$ $(\eta \in {}^{<n}2)$ and $D(\eta) \in S_1(M, \{a(\zeta) : \zeta \subsetneqq \eta\})$ $(\eta \in {}^{\leq n}2)$ such that $D(\eta) \supseteq D(\eta^\frown\langle j\rangle)$ $(j < 2)$ and $D(\eta^\frown\langle 0\rangle) \neq D(\eta^\frown\langle 1\rangle)$ for all $\eta \in {}^{<n}2$, and $D(\eta) \cap D \neq \varnothing$ for all $\eta \in {}^{\leq n}2$. As previously stated, by convention $r_M(D) = -1$ if $D = \varnothing$.

Let $D_i(\eta)$ $(\eta \in {}^{\leq i}2)$ be subsets of M_i and $a_i(\eta)$ $(\eta \in {}^{<i}2)$ be elements of M_i witnessing that

$$r_{M_i}(\{a \in C_i : \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \supseteq B_i\}) \geqq i.$$

Towards a contradiction suppose there exists infinite $Z \subseteq \omega$ such that $d(\mathcal{F}_i)$ is finite and bounded for $i \in Z$. Then, for $i \in Z$, if E_i is the 0-definable equivalence relation on C_i defined by $\mathrm{cl}^{\mathcal{F}_i}(\{x\}) = \mathrm{cl}^{\mathcal{F}_i}(\{y\})$, the quotient C_i/E_i has bounded size. Therefore, by 4.1, if C'_i is an E_i-class, then $r_M(C'_i) \to \omega$ as $i \to \omega$. Thus d_0 can be increased. From this contradiction we infer that $d(\mathcal{F}_i) \to \omega$ as $i \to \omega$.

Let $A_i(j)$ denote $\{a_i(\eta) : \eta \in {}^{<j}2\}$. Consider a very large value of i. Then $d(\mathcal{F}_i)$ is also very large. From 4.4, $\mathrm{cl}_{M_i}^{\mathcal{F}}(A_i(r_1))$ exists with size bounded in terms of r_1. Let B'_i be a complete set of representatives for $\mathcal{F}$ disjoint from $B_i \cup \mathrm{cl}_{M_i}^{\mathcal{F}}(A_i(r_1))$. From (x) there exists $\eta \in {}^{r_1}2$ such that

$$D_i(\eta) \cap \{a \in C_i : \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \supseteq B_i \ \& \ \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \cap B'_i \neq \varnothing\} = \varnothing.$$

Therefore

$$D_i(\eta) \cap \{a \in C_i : \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \supseteq B_i\} \subseteq \{a \in C_i : \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \subseteq B_i \cup \mathrm{cl}_{M_i}^{\mathcal{F}}(A_i(r_1))\}.$$

Recalling that $|\mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\})| > |B_i|$, we see that

$$r_{M_i}(\{a \in C_i : \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \supseteq B_i \ \& \ \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \cap \mathrm{cl}_{M_i}^{\mathcal{F}}(A_i(r_1)) \neq \varnothing\}) \geqq i - r_1.$$

Since i is large, by 4.1 there exists $b_i \in \mathrm{cl}_{M_i}^{\mathcal{F}}(A_i(r_1))$ such that

$$r_{M_i}(\{a \in C_i : \mathrm{cl}_{C_i}^{\mathcal{F}}(\{a\}) \supseteq B_i \cup \{b_i\}\}) \geqq (i - r_1 - j)/j$$

where j is a bound on $|\mathrm{cl}_{M_i}^{\mathcal{F}}(A_i(r_1))|$ computed from r_1. It is clear that $\boldsymbol{S}'$ could have been chosen to make $|B_i| = m + 1$ for all $i < \omega$. This contradicts the choice of m, so the proof is complete.

Lemma 4.6. *If $M \in \boldsymbol{H}(L)$, $a \in P \in S_1(M, \varnothing)$ and $Q \in S_2(M, \varnothing)$, then $r_M(B) \leqq r_M(P) + r_M(B_a)$, where B denotes the set $\{b \in M : \exists x'(x' \in P \ \& \ \langle x', b\rangle \in Q)\}$ and B_a denotes $\{b \in M : \langle a, b\rangle \in Q\}$.*

Proof. We proceed by induction on $r_M(P) + r_M(B_a)$. Choose $c \in M$ such that there are distinct $B_0, B_1 \in S(\{c\})$ with $B_0, B_1 \subseteq B$ and $r_M(B_i) \geqq r_M(B) - 1$ for $i < 2$. Choose $P_i \in S_1(\{c\})$ $(i < 2)$ such that for all $b \in B_i$ there exists $a \in P_i$ such that $\langle a, b\rangle \in Q$. For $i < 2$ and $a \in P_i$, let $B_{i,a}$ denote $B_a \cap B_i$. Let $M' = (M, \{c\})$.

Suppose $P_0 = P_1$ and let $a \in P_0 = P_1$. There exists $j < 2$ such that $r_M(B_{j,a}) < r_M(B_a)$. By 4.3(ii), $r_{M'}(B_j) = r_M(B_j)$, $r_{M'}(B_{j,a}) = r_M(B_{j,a})$, and $r_{M'}(P_j) = r_M(P_j)$. By the induction hypothesis

$$r_M(B_j) = r_{M'}(B_j) \leqq r_{M'}(P_j) + r_{M'}(B_{j,a}) < r_M(P) + r_M(B_a).$$

From this and the earlier inequality for $r_M(B)$

$$r_M(B) \leqq r_M(B_j) + 1 \leqq r_M(P) + r_M(B_a).$$

Suppose $P_0 \neq P_1$. There exists $j < 2$ such that $r_M(P_j) < r_M(P)$. Let $a \in P_j$. Notice that $r_{M'}(P_j) = r_M(P_j)$ and $r_{M'}(B_{j,a}) \leqq r_M(B_a)$. By the induction hypothesis

$$r_M(B_j) + r_{M'}(B_j) \leqq r_{M'}(P_j) + r_{M'}(B_{j,a}) < r_M(P) + r_M(B_a).$$

As before this yields the desired conclusion. This completes the proof of the lemma.

Lemma 4.7. *Let r_0, $m < \omega$ and $r(M) < r_0$ for all transitive $M \in \boldsymbol{H}(L)$ such that $|s_2(M, \emptyset)| < m$. There exists $r_1 < \omega$ such that, if $M \in \boldsymbol{H}(L)$ is transitive, $|s_2(M, \emptyset)| = m$, and $\mathscr{E}(M)$ contains a nontrivial member, then $r(M) < r_1$.*

Proof. We shall consider quintuples $\langle M, A, N, E, j \rangle$ such that $M \in \boldsymbol{H}(L)$ is transitive, $s_2(M, \emptyset) = m$, $A \subseteq M$ is finite, $N \in S_1(A)$, $E \in \mathscr{E}(N)$ is nontrivial, and

$$j = j(N, E) = |\{\mathrm{tp}(b, c) : b, c \in N \;\&\; b \not E c\}|.$$

Let $\omega \times \omega$ be ordered lexicographically. By induction on $\langle j, r_M(C) \rangle$, where C is an E-class, we will prove that there is a uniform finite bound for $r_M(N)$.

Fix $a \in C$.

Case 1. For all $b, c \in N$ if $a \not E b$, $a \not E c$, and $b E c$, then $\mathrm{tp}(a, b) = \mathrm{tp}(a, c)$. Let N_0 be obtained by selecting one element from each E-class. Then N_0 does not depend on the choices and $N_0 \in \boldsymbol{H}(L)$ since $N \in \boldsymbol{H}(L)$ and L is binary. Let N_1 be an E-class seen as an L-structure. Clearly $N_1 \in \boldsymbol{H}(L)$ also. Thus we have $N = N_0[N_1]$ and $|s_2(N_0, \emptyset)|$, $|s_2(N_1, \emptyset)| < m$. Therefore $r_M(N)$ can be bounded by 4.2 and 4.5.

Case 2. Otherwise. There exist $b, c \in N$ such that $a \not E b$, $a \not E c$, $b E c$, and $\mathrm{tp}(a, b) \neq \mathrm{tp}(a, c)$. Let N_b, N_c be the solution sets of the types $\mathrm{tp}(b \mid A \cup \{a\})$, $\mathrm{tp}(c \mid A \cup \{a\})$. Let E_b, E_c be the equivalence relations on N_b, N_c induced by E. Let C_b be the E_b-class containing b and C_c be the E_c-class containing c. One of C_b, C_c has rank less than $r_M(C)$, say $r_M(C_b) < r_M(C)$. Since $j(N_b, E_b) \leqq j$, by the induction hypothesis $r_M(N_b)$ can be bounded.

If C is an E-class, $|s_2(C, \emptyset)| < m$. Therefore $r(C) < m$ by hypothesis. By 4.5 we have a bound for $r_M(C)$.

Let

$$\mathscr{Q} = \{\mathrm{tp}(a, b)\} \cup \{\mathrm{tp}(a, d) : d \in N,\ d \neq a, \text{ and } d E a\}.$$

From above, if $q \in \mathcal{Q}$ and $B_a(q) = \{d \in N : \mathrm{tp}(a, d) = q\}$, then $r_M(B_a(q))$ is bounded. For $P \in S_1(N, \{a\})$ and $q \in \mathcal{Q}$, let $A(P, q) = \{e \in N : (\exists d \in P)(\mathrm{tp}(d, e) = q)\}$ and $S(P) = \bigcup\{A(P, q) : q \in \mathcal{Q}\}$. Let $\mathcal{P}$ be the least subset of $S_1(N, \{a\})$ including $\{a\}$ and stable under S. Clearly, for each $P \in \mathcal{P}$ there exists a finite sequence $\langle P_0, \ldots, P_k \rangle \in \mathcal{P}$ such that $P_{i+1} \in S(P_i)$ and one can easily arrange for the P_i's to be distinct. Moreover, for each $i < k$, there exists $q \in \mathcal{Q}$ such that for all $e \in P_{i+1}$ there exists $d \in P_i$ with $\mathrm{tp}(d, e) = q$. Now $k < |s_2(N, \emptyset)| \leqq |s_2(M, \emptyset)|$ and so k is bounded since L is fixed. By induction on i using 4.6 we can bound $r_{(M,a)}(P_i)$ for all $i < k$. From 4.3, $r_M(P_i) = r_{(M,a)}(P_i)$ and so by 4.1 we can compute a bound for $r_M(\bigcup \mathcal{P})$ in terms of L.

Let $\bigcup \mathcal{P}$ be denoted B_a and notice that B_a is $\{a\}$-definable in N. In fact, B_a consists of all $d \in N$ such that there exists a finite sequence $\langle a_0, \ldots, a_k \rangle$ in N such that $a_0 = a$, $a_k = d$, and $\mathrm{tp}(a_i, a_{i+1}) \in \mathcal{Q}$ for each $i < k$. It follows that if $d \in B_a$, then $B_d \subseteq B_a$. If $B_d \subsetneqq B_a$, then we can find a strictly decreasing ω-sequence of uniformly definable subsets of N, which contradicts the stability of M. Therefore $B_d = B_a$, which means that there exists $\mathscr{E}' \in E(N)$ such that $B_a = \bigcup \mathcal{P}$ is an E'-class. Let this E'-class be denoted C'. By choice of $\mathcal{Q}$, $\bigcup \mathcal{P} \supseteq C \cup N_b$. Hence $C \subsetneqq C'$. Thus $j(N, E') < j(N, E)$. Either $C' = N$ and we have a bound for $r_M(N)$ since we have one for $r_M(\bigcup \mathcal{P})$, or E' is nontrivial and we have a bound for $r_M(N)$ by the induction hypothesis.

If $\mathscr{E}(M)$ contains a nontrivial member E, then there is a quintuple $\langle M, \emptyset, M, E, j \rangle$ and the above discussion allows us to find a bound for $r(M)$ in terms of L.

LEMMA 4.8. *Let r_0, $m < \omega$ and $r(M) < r_0$ for all transitive $M \in \boldsymbol{H}(L)$ such that $|s_2(M, \emptyset)| < m$. There exists $F : \omega \to \omega$ depending only on L and m such that, if $a \in M \in \boldsymbol{H}(L)$, M is transitive, and $|s_2(M, \emptyset)| = m$, then*

$$r(M) < F(\min\{r(P) : P \in S_1(M, \{a\}) - \{\{a\}\}\}).$$

PROOF. Let $a \in M \in \boldsymbol{H}(L)$, M be transitive, $|s_2(M, \emptyset)| = m$, and $P \in S_1(M, \{a\}) - \{\{a\}\}$. We shall show that either $r(M)$ is bounded outright or bounded in terms of $r_M(P)$. Since $r_M(P)$ can be bounded in terms of $r(P)$ by 4.5, this will be enough to prove the lemma.

Let $q \in s_2(M, \emptyset)$ be the 2-type such that for $b \in M$, $b \in P$ if $\mathrm{tp}(a, b) = q$. Let $\mathcal{P}$ be the least subset of $S_1(N, \{a\})$ such that $\{a\} \in \mathcal{P}$ and for all $P_0 \in \mathcal{P}$ and $P_1 \in S_1(N, \{a\}) - \mathcal{P}$ if $d \in P_0$ and $e \in P_1$, then $\mathrm{tp}(d, e) \neq q$. Arguing as in the proof of 4.7 with the role of $\mathcal{Q}$ being played by $\{q\}$ we can compute a bound for $r_M(\bigcup \mathcal{P})$ in terms of $r_M(P)$. We also see that $\bigcup \mathcal{P}$ is an E-class for some

$E \in \mathscr{E}(M)$. If E is nontrivial, we have a bound for $r(M)$ from 4.7. Otherwise, $\bigcup \mathscr{P} = M$ and as already noted we have a bound for $r(M)$ in terms of $r_M(P)$. This completes the proof.

5. The key reduction

Our aim is to show that for a fixed finite relational language L there exists $r < \omega$ such that $r(M) < r$ for all $M \in \boldsymbol{H}(L)$. From 4.1 and 4.5 it is sufficient to consider the case in which M is transitive. For transitive M we proceed by induction on $|s_2(M, \varnothing)|$. Towards a contradiction let $m < \omega$ be the least number such that $r(M)$ can be arbitrarily large while $|s_2(M, \varnothing)| = m$. From 4.7 there is a finite bound on $r(M)$ for those M with $|s_2(M, \varnothing)| = m$ which admit a nontrivial 0-definable equivalence relation.

The next step in the proof is the crucial one. Suppose we have M with $|s_2(M, \varnothing)| = m$ and $r(M)$ so large that M admits no nontrivial 0-definable equivalence relation, then assuming that $|M|$ is also large enough there are $B \subseteq M$ and $N \in S_1(M, B)$ with $|B| = 2$, $|N|$ large, and $|s_2(N, \varnothing)| < m$.

We suppose that L is fixed throughout this section and as in §3 and §4 we do not mention explicitly the dependence of various functions and bounds on L.

Lemma 5.1. *There exists $F : \omega \to \omega$ such that, if $M \in \boldsymbol{H}(L)$ is transitive, $i < \omega$, and $|M| > F(i)$, then either M is an indiscernible set, or $\mathscr{E}(M)$ contains a nontrivial member, or there exists $B \subseteq M$ and $N \in S_1(M, B)$ such that $|B| = 2$, $|N| > i$, and $|s_2(N, \varnothing)| < |s_2(M, \varnothing)|$.*

Proof. From 7.4 of [2], if M is infinite and σ is an L-sentence true in M, there exists finite $M' \subseteq M$ such that $M' \in \boldsymbol{H}(L)$ and $M' \models \sigma$. Hence for the proof of the lemma we may assume that M is finite.

Suppose that $\mathscr{E}(M)$ contains no nontrivial member and that M is not indiscernible. For $b_0, b_1 \in M$ define

$$A(b_0, b_1) = \{a \in M : \mathrm{tp}(a, b_0) = \mathrm{tp}(a, b_1)\}.$$

Since M is stable, if $\mathrm{tp}(c_0, c_1) = \mathrm{tp}(b_0, b_1)$ we cannot have $A(b_0, b_1) \subsetneq A(c_0, c_1)$. Thus we can fix distinct b_0 and b_1 such that $A(b_0, b_1)$ is maximal in $\{A(c_0, c_1) : c_0, c_1 \in M, c_0 \neq c_1\}$. Let $M - A(b_0, b_1)$ be denoted $A'(b_0, b_1)$.

Claim. *If $n < \omega$ and $|M|$ is sufficiently large compared with n, then $|A'(b_0, b_1)| > n$.*

Proof. Suppose that $|M|$ is very large and towards a contradiction that $|A'(b_0, b_1)| < n$. Let $p = \mathrm{tp}(b_0, b_1)$. Since $\mathscr{E}(M)$ contains no nontrivial member, for any $d_0, d_1 \in M$ there exists a finite sequence $\langle c_0, \ldots, c_k \rangle \in M$ such that $d_0 = c_0$, $c_k = d_1$, and $\mathrm{tp}(c_i, c_{i+1}) = p$ for each $i < k$. Since L is fixed, there exists $K < \omega$ such that $\langle c_0, \ldots, c_k \rangle$ can always be found with $k < K$. From the definition of $A'(d_0, d_1)$ we have

$$A'(d_0, d_1) \subseteq \bigcup \{A'(c_i, c_{i+1}) : i < k\}.$$

Hence

$$|A'(d_0, d_1)| \leqq k \cdot |A'(b_0, b_1)| < Kn.$$

The bound K will work equally well for any nontrivial $p' \in s_2(M, \emptyset)$. For $a \in M$ define

$$A(a, p') = \{c \in M : \mathrm{tp}(a, c) = p'\}.$$

Let $|A(a, p')|$, which depends only on p', be denoted $k(p')$. Since every $b \in M$ can be reached from a in $\leqq K$ "p'-steps", we have

$$1 + k(p') + k(p')^2 + \cdots + k(p')^K \geqq |M|.$$

Therefore $k(p')$ is large for every nontrivial $p' \in s_2(M, \emptyset)$. Choose distinct $c_0, \ldots, c_{Kn} \in A(b_0, p)$. Since M is nontrivial there exists $p' \in s_2(M, \emptyset)$ such that $p' \neq \mathrm{tp}(b_0, b_1)$. Fix such p'. For each $i \leqq Kn$, $|A(c_i, p')| = k(p')$ is large. For all $i, j \leqq Kn$, $A(c_i, p') - A(c_j, p') \subseteq A'(c_i, c_j)$ and so $|A(c_i, p') - A(c_j, p')|$ is small. Hence there exists $c \in \bigcap \{A(c_i, p') : i \leqq Kn\}$ and $c_i \in A'(b_0, c)$ for all $i \leqq Kn$ by choice of p'. Thus $|A'(b_0, c)| > Kn$. This contradiction completes the proof of the Claim.

Returning to the proof of the theorem, define

$$R = \{\langle a_0, a_1 \rangle \in M^2 : \mathrm{tp}(a_0 \,|\, A(b_0, b_1)) = \mathrm{tp}(a_1 \,|\, A(b_0, b_1))\}.$$

R is clearly an equivalence relation on M. Since M is finite there exists $\alpha \in \mathrm{aut}(M)$ such that $\alpha(b_0) = b_1$ and $A(b_0, b_1) \subseteq \mathrm{fix}(\alpha)$. If $c \in A'(b_0, b_1)$, then $\alpha(c) \neq c$ since $\mathrm{tp}(b_0, c) \neq \mathrm{tp}(b_1, c)$. Therefore $|c/R| > 1$ iff $c \in A'(b_0, b_1)$. By the maximality of $A(b_0, b_1)$, if cRd and $c \neq d$, then $A(c, d) = A(b_0, b_1)$.

Let $B = \{b_0, b_1\}$. If $|M|$ is large enough compared with n, then by the Claim there exists $N \in S_1(M, B)$ such that $|N| > n$ and $N \subseteq A'(b_0, b_1)$. Fix $q \in s_2(M, \emptyset)$ which is nontrivial such that $\mathrm{tp}(c, d) = q$ for some $c \in N$ and $d \in M$ such that cRd. Then for every $c \in N$ there exists a unique $d \in M$ such that cRd and $\mathrm{tp}(c, d) = q$. (If there were distinct d_0, d_1 in the R-class of c such that $\mathrm{tp}(c, d_0) = \mathrm{tp}(c, d_1)$, then $c \in A(d_0, d_1)$ contradicting our finding that $A(d_0, d_1) =$

$A(b_0, b_1)$.) For any $c \in N$ let $q(c)$ denote the unique y such that $\mathrm{tp}(c, y) = q$ and cRy. Let $N' = \{q(x) : x \in N\}$. Notice that $N' \in S_1(M, B)$. For any $d \in N'$, let $\hat{q}(d)$ denote the unique $x \in N$ such that $\mathrm{tp}(x, d) = q$ and xRd. Let c vary in N and d in N'. If $\mathrm{tp}(c, d) = q$, then $\mathrm{tp}(c, d \mid B) = \mathrm{tp}(c, q(c) \mid B)$ since L is binary. But $\{q(c)\}$ is $B \cup \{c\}$-definable since R is B-definable. Thus $\mathrm{tp}(c, d) = q$ implies $d = q(c)$. Notice also that each R-class contains at most one member of N by the maximality of $A(b_0, b_1)$.

Let $c, e \in N$ and $c \neq e$. Then $c \not R e$ and there exists $d = q(e) \in N'$ such that dRe, whence $c \not R d$. The number of possibilities for $\mathrm{tp}(c, d)$ is less than $|s_2(M, \emptyset)| - 1$ because $\mathrm{tp}(c, d) \neq q$. But $\mathrm{tp}(c, d)$ fixes $\mathrm{tp}(c, e)$ since $e = \hat{q}(d)$. Therefore the number of possibilities for $\mathrm{tp}(c, e)$ is less than $|s_2(M, \emptyset)| - 1$. This shows that $|s_2(N, \emptyset)| < |s_2(M, \emptyset)|$ and completes the proof of the lemma.

We now continue the discussion begun at the beginning of the section. In 5.1 we have seen that it is sufficient to examine $M \in \boldsymbol{H}(L)$ such that $|s_2(M, \emptyset)| = m$ and M has a large subset N defined from a pair of elements of M with $r(N)$ bounded. Our next task is to exploit the existence of N.

Let $M \in \boldsymbol{H}(L)$ be transitive and $p \in s_2(M, \emptyset)$ be a nontrivial 2-type. For $n < \omega$ define $M[p, n]$ by induction on n as follows. Let $M[p, 0] = M$. Suppose $M[p, n] \subseteq M$ has been found. If $M[p, n] = \emptyset$, let $M[p, n+1] = \emptyset$. If $M[p, n] \neq \emptyset$, choose $a_n \in M[p, n]$ and define

$$M[p, n+1] = \{a \in M[p, n] : \mathrm{tp}_M(a_n, a) = p\}.$$

Since L is binary $\mathrm{tp}_M(a_0, \ldots, a_n)$ is fixed by p and n whence the isomorphism type of $M[p, n]$ is fixed by p and n. Notice that, if $M[p, n] \neq \emptyset$, then $M[p, n] \in \boldsymbol{H}(L)$ and is transitive.

LEMMA 5.2. *There exists $F : \omega^2 \to \omega$ such that, if $M \in \boldsymbol{H}(L)$ is transitive, $B \subseteq M$ is finite, $N \in S_1(M, B)$, and $|N| \geqq F(|B|, r(N))$, then there exists nontrivial $p \in s_2(M, \emptyset)$ such that $r(M[p, F(|B|, r(N))]) \leqq r(N)$.*

PROOF. Assume that $|N|$ is very large compared with $|B|$ and $r(N)$. Let $\bar{b}$ enumerate B. We shall find a 2-type p such that $r(M[p, j]) \leqq r(N)$ with j bounded in terms of $|B|$ and $r(N)$.

From 3.7 there exist $\bar{a} \in N$, $A = \mathrm{rng}(\bar{a})$, $P(\bar{a}, \bar{b}) \in S_1(N, A)$, and $E(\bar{a}, \bar{b}) \in \mathscr{E}((N, A))$ such that

(i) $|A| = l(\bar{a})$ is bounded in terms of $r(N)$,

(ii) $\mathrm{fld}(E(\bar{a}, \bar{b})) = P(\bar{a}, \bar{b})$ and $|P(\bar{a}, \bar{b})/E(\bar{a}, \bar{b})|$ is large compared with $|B|$, $r(N)$, and h where h is the cardinality of the $E(\bar{a}, \bar{b})$-classes,

(iii) if $n < \omega$ and $c_0, \dots, c_n \in P(\bar{a}, \bar{b})$ fall in distinct $E(\bar{a}, \bar{b})$-classes then $\text{tp}(c_0, \dots, c_n)$ depends only on n.

The idea of the proof is as follows. We shall show that there exists m bounded in terms of $|B|$ and $r(N)$ such that $M[p, m]$ is almost isomorphic to $P(\bar{a}, \bar{b})$ viewed as a structure, the only possible difference between the two structures being that one may have more equivalence classes than the other. For both structures the number of equivalence classes is relatively large. Thus $r(M[p, m]) = r(P(\bar{a}, \bar{b})) \leqq r(N)$.

Let Q be the set of tuples in M conjugate to $\bar{a} {}^\frown \bar{b}$. Let $k = l(\bar{a} {}^\frown \bar{b})$ and m be a bound on $|s_{2k}(M, \emptyset)|$ computed from $|B|$ and $r(N)$. Then m is small compared with $|P(\bar{a}, \bar{b})/E(\bar{a}, \bar{b})|$. Since L is binary, if $\bar{d} \in Q$, then

$$|\{\text{tp}(\bar{d}_0, \bar{d}_1 \mid A \cup B) : \bar{d}_0, \bar{d}_1 \text{ are conjugates of } \bar{d} \text{ over } A \cup B\}| \leqq m.$$

Let $\bar{d} \in Q$.

By $P(\bar{d})$ we denote the image of $P(\bar{a}, \bar{b})$ under an automorphism of M which maps $\bar{a} {}^\frown \bar{b}$ into $\bar{d}$.

Claim. *If $P(\bar{d})$ intersects at least m of the $E(\bar{a}, \bar{b})$-classes, then there are fewer than m $E(\bar{a}, \bar{b})$-classes which are not included in $P(\bar{d})$.*

Proof of Claim. We first note that there are fewer than m $E(\bar{a}, \bar{b})$-classes disjoint from $P(\bar{d})$. This follows from the observation that, for $\bar{d}_0$, $\bar{d}_1$ conjugate to $\bar{d}$ over $A \cup B$, $|(P(\bar{d}_0)/E(\bar{a}, \bar{b})) - (P(\bar{d}_1)/E(\bar{a}, \bar{b}))|$ could take any value $\leqq m$ were there m $E(\bar{a}, \bar{b})$-classes disjoint from $P(\bar{d})$. Next observe that M is prime over $A \cup B \cup P(\bar{a}, \bar{b})$ since $P(\bar{a}, \bar{b}) \in S_1(M, A \cup B)$. From (iii), if $\pi \in \text{perm}(P(\bar{a}, \bar{b}))$ and $\pi \restriction C \in \text{aut}(C)$ for each $C \in P(\bar{a}, \bar{b})/E(\bar{a}, \bar{b})$, then π is an elementary map. Thus any such π can be extended to $\alpha \in \text{aut}(M)$. Since M is transitive, if C is an $E(\bar{a}, \bar{b})$-class such that $P(\bar{d}) \cap C \neq \emptyset$ and $C \not\subseteq P(\bar{d})$, then there exists $\gamma \in \text{aut}(C)$ with $P(\bar{d}) \cap C \neq P(\bar{d}) \cap \gamma(C)$. Now we see that, if there are m $E(\bar{a}, \bar{b})$-classes X such that $\emptyset \neq X \cap P(\bar{d}) \neq X$, then

$$|\{X \in P(\bar{a}, \bar{b})/E(\bar{a}, \bar{b}) : X \cap P(\bar{d}_0) \neq X \cap P(\bar{d}_1)\}|$$

can take any value $\leqq m$ with $\bar{d}_0$, $\bar{d}_1$ conjugate to $\bar{d}$ over $A \cup B$. Thus $P(\bar{d}) \supseteq X$ for almost all $E(\bar{a}, \bar{b})$-classes X. Hence, if there were $\geqq m$ $E(\bar{a}, \bar{b})$-classes $X \not\subseteq P(\bar{d})$, then $|(P(\bar{d}_0) - P(\bar{d}_1))/E(\bar{a}, \bar{b})|$ could take any value $\leqq m$ with $\bar{d}_0$, $\bar{d}_1$ conjugate to $\bar{d}$ over $A \cup B$. The latter is impossible since there are $\leqq m$ possibilities for $\text{tp}(\bar{d}_0, \bar{d}_1 \mid A \cup B)$. This completes the proof of the claim.

Recall that $|P(\bar{a}, \bar{b})/E(\bar{a}, \bar{b})|$ is large compared with $|B|$, $r(N)$, and h, and hence also large compared with $|A|$ and m. We conclude that, if $\bar{d}_0, \bar{d}_1 \in Q$ and $|P(\bar{d}_0) \cap P(\bar{d}_1)| > (m-1)h$, then $|P(\bar{d}_0) - P(\bar{d}_1)| < mh$. Therefore

$$E_Q = \{\langle \bar{d}_0, \bar{d}_1 \rangle : \bar{d}_0, \bar{d}_1 \in Q \text{ and } |P(\bar{d}_0) \cap P(\bar{d}_1)| > (m-1)h\}$$

is an equivalence relation on Q.

For an application of 3.9 in (M^*, C_Q) let $R(x,y)$ mean "$x \in M$, $y \in Q$, $yE_Q\bar{a}^\frown\bar{b}$, and $x \in P(y)$."

Let C_Q denote the E_Q-class containing $\bar{a}^\frown\bar{b}$. By 3.9 there is a $\{C_Q\}$-definable set $P^\square(\bar{a},\bar{b})$ whose symmetric difference with $P(\bar{a},\bar{b})$ is bounded in terms of m and h. Let $c_1,\dots,c_m \in P(\bar{a},\bar{b})$ be $E(\bar{a},\bar{b})$-inequivalent. From (iii) there exists $p \in s_2(M,\emptyset)$ such that $\operatorname{tp}(c_i,c_j) = p$ for all i,j such that $1 \leqq i$, $j \leqq k$ and $i \neq j$. Since $\{C_Q\}$ is $\{c_1,\dots,c_m\}$-definable, so is $P^\square(\bar{a},\bar{b})$. Let $P'(c_1,\dots,c_m)$ denote

$$\{c \in M : (\forall i)(1 \leqq i \leqq m \rightarrow \operatorname{tp}(c_i,c) = p)\}.$$

Thus $P'(c_1,\dots,c_m) = M[p,m]$ and $P' \in S_1(M,\{c_1,\dots,c_m\})$.

Since $P(\bar{a},\bar{b})/E(\bar{a},\bar{b})$ is large compared with m and h there exists $c \in P(\bar{a},\bar{b}) \cap P^\square(\bar{a},\bar{b})$ such that c is not $E(\bar{a},\bar{b})$-equivalent to any of $c_1,\dots,c_m$. Clearly $c \in P'(c_1,\dots,c_m)$, whence $P'(c_1,\dots,c_m) \subseteq P^\square(\bar{a},\bar{b})$ and also $|P(\bar{a},\bar{b}) - P'(c_1,\dots,c_m)| \leqq mh$. Let

$$E = \{\langle x,y \rangle : x,y \in P'(c_1,\dots,c_m) \ \&\ \operatorname{tp}(x,y) \neq p\}.$$

Then E agrees with $E(\bar{a},\bar{b})$ on

$$P(\bar{a},\bar{b}) - \bigcup\{c_i/E(\bar{a},\bar{b}) : 1 \leqq i \leqq m\} \subseteq P'(c_1,\dots,c_m).$$

Notice that $P'(c_1,\dots,c_m)$ is a transitive structure and that $|P'(c_1,\dots,c_m) - P(\bar{a},\bar{b})|$ is small compared with $|P(\bar{a},\bar{b})/E(\bar{a},\bar{b})|$. Hence for each $c \in P'(c_1,\dots,c_m)$, $|\{d : cEd\}|$ is small compared with $|P(\bar{a},\bar{b})/E(\bar{a},\bar{b})|$. It follows that there is an $E(\bar{a},\bar{b})$-class $C \subseteq P'(c_1,\dots,c_m)$ such that no $d \in P'(c_1,\dots,c_m) - C$ is E-related to $c \in C$. Therefore E is an equivalence relation on $P'(c_1,\dots,c_m)$ and the E-classes are all isomorphic to $E(\bar{a},\bar{b})$-classes, where E-classes and $E(\bar{a},\bar{b})$-classes are being viewed as structures in their own right. (Of course, most of the E-classes are $E(\bar{a},\bar{b})$-classes.) Further, since any two $E(\bar{a},\bar{b})$-classes are p-related and since almost every E-class is an $E(\bar{a},\bar{b})$-class and $P'(c_1,\dots,c_m)$ is transitive, we see that any two E-classes are p-related, i.e., if $c,d \in P'(c_1,\dots,c_m)$ are not E-related, then $\operatorname{tp}(c,d) = p$.

Let us compare the structures $P'(c_1,\dots,c_m)$ and $P(\bar{a},\bar{b})$. These are both transitive structures in $\boldsymbol{H}(L)$, and the same L-formula defines E on $P'(c_1,\dots,c_m)$ as defines $E(\bar{a},\bar{b})$ on $P(\bar{a},\bar{b})$. Moreover, each E-class is isomorphic to each $E(\bar{a},\bar{b})$-class; and if x, y in one of the structures are not in the same equivalence class, then the quantifier-free type of $\langle x, y\rangle$ depends neither on choice of x, y nor on which of the two structures we are looking at. Thus the structures differ only in that $|P'(c_1,\ldots,c_m)/E|$ and $|P(\bar{a},\bar{b})/E(\bar{a},\bar{b})|$ may not be equal. However, both of these "dimensions" are large compared with $r(N)$ and $P(\bar{a},\bar{b})\subseteq N$. Thus

$$r(P'(c_1,\ldots,c_m)) = r(P(\bar{a},\bar{b}))\leqq r(N).$$

Since $P'(c_1,\cdots,c_m) = M[p,m]$ and m is bounded in terms of $|B|$ and $r(N)$, the proof is complete.

6. Conclusion

We are now ready to prove the main result of the paper.

THEOREM 6.1. *Let L be a finite binary relational language. There exists $r<\omega$ such that $r(M)<r$ for all $M\in \boldsymbol{H}(L)$.*

PROOF. From L we can compute a bound on $|s_2(M,\varnothing)|$ for $M\in\boldsymbol{H}(L)$. Towards a contradiction suppose m is the least number such that

$$\{r(M): M\in \boldsymbol{H}(L), M \text{ transitive, and } |s_2(M,\varnothing)| = m\}$$

has no finite bound. Let

$$r_0 = \max\{r(M): M\in \boldsymbol{H}(L), M \text{ transitive, and } |s_2(M,\varnothing)| < m\}.$$

Consider transitive $M\in\boldsymbol{H}(L)$ with $|s_2(M,\varnothing)| = m$ such that $r(M)$ is very large. Clearly $|M|$ is also large. From 4.7 $\mathscr{E}(M)$ has no nontrivial member. From 5.1 there exist $B\subseteq M$ and $N\in S_1(M,B)$ such that $|B|=2$, $|N|$ is large, and $|s_2(N,\varnothing)|<|s_2(M,\varnothing)|$. Clearly $r(N)\leqq r_0$. Without loss of generality we can assume that the function $F:\omega^2\to\omega$ from 5.2 is increasing in its second argument. Let $i_0 = F(2,r_0)$. By 5.2 there exist nontrivial $p\in s_2(M,\varnothing)$ such that $r(M[p,i_0])\leqq r_0$. If $M[p,i_0]=\varnothing$, let j_0 be the greatest number such that $M[p,j_0]\neq\varnothing$. Otherwise, let $j_0=i_0$. If $j_0<i_0$, then $|s_2(M[p,j_0])|<|s_2(M,\varnothing)|$. Thus, whatever the value of j_0, $r(M[p,j_0])\leqq r_0$. From the definition of $M[p,n]$ in §5 it is clear that, if $M[p,n+1]\neq\varnothing$, then $M[p,n+1]$ bears exactly the same relation to $M[p,n]$ that $M[p,1]$ bears to $M[p,0]$, i.e., there exists nontrivial $p_n\in s_2(M[p,n],\varnothing)$ such that $M[p,n+1] = (M[p,n])[p_n,1]$. Let $F:\omega\to\omega$ be the function from 4.8. Without loss of generality suppose that F is increasing. From 4.8 for each $n<j_0$

$$r(M[p,n]) < F(\min\{r(P): P \in S_1(M[p,n],\{a_n\}) - \{\{a_n\}\}\}),$$

where a_n is the element used to define $M[p,n+1]$. Since L is binary, for $a,b,c \in M[p,n]$,

$$\mathrm{tp}_{M[p,n]}(a,b) = \mathrm{tp}_{M[p,n]}(a,c) \Leftrightarrow \mathrm{tp}_M(a,b) = \mathrm{tp}_M(a,c).$$

Therefore $M[p,n+1] \in S_1(M[p,n],\{a_n\}) - \{\{a_n\}\}$, and so for each $n < j_0$

$$r(M[p,n]) < F(r(M[p,n+1])).$$

Therefore $r(M) < F^{j_0}(r_0)$ where F^{j_0} denotes the j_0-th iterate of F. This contradicts our assumption that $r(M)$ is very large and shows that there is a finite bound on $r(M)$ as M runs through $\boldsymbol{H}(L)$, at least for transitive M. But as already observed from 4.1 and 4.5 the bound on $r(M)$ for transitive M yields a finite bound on $r(M)$ for all $M \in \boldsymbol{H}(L)$.

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Institute for Advanced Studies
The Hebrew University of Jerusalem
Jerusalem Israel

Current address of first author
Department of Mathematics
Simon Fraser University
Burnaby, B.C., Canada V5A 1S6

Current address of second author
Institute of Mathematics
The Hebrew University of Mathematics
Jerusalem, Israel